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1979 J. Phys. A: Math. Gen. 12 L151

(http://iopscience.iop.org/0305-4470/12/7/001)

J. Phys. A: Math. Gen., Vol. 12, No. 7, 1979. Printed in Great Britain

LETTER TO THE EDITOR

Topology of textures in the planar phase of superfluid ^{3}He

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Received 29 January 1979

Abstract. Singular and non-singular textures of the planar phase of superfluid ^{3}He are classified using topological homotopy group methods. The effects of an applied magnetic field and of boundary conditions at the walls of the container are discussed.

The dipolar interaction between ^{3}He atoms is expected to stabilise a superfluid phase called the planar phase over a narrow temperature interval ($\approx 10^{-6}\ T_c$) between the normal phase and the A or B phase (Leggett 1975, Jones *et al* 1976). Topological homotopy group methods have been introduced into condensed-state physics by Toulouse and Kleman (1976) and Volovik and Mineev (1977a, b). In this Letter we use these methods to classify the textures which may be seen in the planar phase when good enough temperature control is obtained to perform experiments on this phase.

The order parameter for any superfluid phase of ^{3}He is a complex 3×3 matrix $A_{\mu i}$ which transforms as a vector under space rotations with respect to the column index i, and as a vector under spin rotations with respect to the row index μ. For the case of the planar phase, the most general order parameter is

$$A_{\mu i} = \Delta\, e^{i\alpha} \epsilon_{\mu i j} n_j \tag{1}$$

where $\boldsymbol{n}$ is a real unit vector and α is a real phase angle, both of which may take different values at different points of space to produce a 'texture', and Δ is a constant magnitude. We shall use the shorthand $(\alpha, \boldsymbol{n})$ for this order parameter.

The topological space of planar phase order parameters is thus

$$R = (S^2 \times S^1)/Z_2. \tag{2}$$

(The quotient with Z_2 corresponds to the fact that $(\alpha, \boldsymbol{n})$ and $(\alpha+\pi, -\boldsymbol{n})$ give the same order parameter $A_{\mu i}$.)

The homotopy groups for R may be calculated from an exact sequence of homomorphisms (see, for example, Steenrod 1951):

$$\pi_2(Z_2)\to\pi_2(S^2\times S^1)\to\pi_2(R)\to\pi_1(Z_2)\to\pi_1(S^2\times S^1)\to\pi_1(R)\to\pi_0(Z_2)\to\pi_0(S^2\times S^1), \tag{3}$$

i.e.

$$0\to Z\to\pi_2(R)\to 0\to Z\to\pi_1(R)\to Z_2\to 0. \tag{4}$$

In this sequence the image of one mapping is the kernel of the next. Consequently,

$$\pi_2(R) = Z \tag{5}$$

and

$$\pi_1(R)/Z = Z_2. \tag{6}$$

Equation (6) still leaves some ambiguity in the identification of $\pi_1(R)$, namely

$$\pi_1(R) = Z + Z_2 \qquad \text{or} \qquad \pi_1(R) = Z.$$

(In the latter case, the Z in the denominator of the quotient is the even integers in $\pi_1(R)$.)

The ambiguity may be resolved by checking directly that two textures of the type

$$(\alpha, \boldsymbol{n}) = (\tfrac{1}{2}\phi, \hat{\boldsymbol{x}} \cos \tfrac{1}{2}\phi + \hat{\boldsymbol{y}} \sin \tfrac{1}{2}\phi)$$

add to a texture of the type

$$(\alpha, \boldsymbol{n}) = (\phi, \hat{\boldsymbol{x}})$$

and *not* to the uniform texture. Thus,

$$\pi_1(R) = Z. \tag{7}$$

The line singularities

$$(\alpha, \boldsymbol{n}) = (m\phi, \hat{\boldsymbol{x}} \cos n\phi + \hat{\boldsymbol{y}} \sin n\phi)$$

and

$$(\alpha, \boldsymbol{n}) = ((m+\tfrac{1}{2})\phi, \hat{\boldsymbol{x}} \cos(n+\tfrac{1}{2})\phi + \hat{\boldsymbol{y}} \sin(n+\tfrac{1}{2})\phi) \tag{8}$$

may be assigned a topological quantum number

$$N(Z) = 2m \qquad \text{or} \qquad 2m+1 \tag{9}$$

respectively. (The situation resembles that which occurs for the A phase of ^{3}He on the surface of a pore of radius small compared with the dipolar length (Bailin and Love 1978a).) These results may also be derived using the algorithm of Mermin (1978).

According to equation (5), point singularities are classified by an integer p, which is the number of times the sphere in order parameter space is covered when a sphere in real space is covered once. Thus the point singularity

$$\boldsymbol{n} = \hat{\boldsymbol{z}} \cos \theta + \sin \theta (\hat{\boldsymbol{x}} \cos p\phi + \hat{\boldsymbol{y}} \sin p\phi) \tag{10}$$

has topological quantum number, p. Following the method of Mermin (1978, see also Volovik and Mineev 1977b) it is easy to check that the action of π_1 on π_2 is trivial, so that different values of p *cannot* be transformed into each other by moving the point singularity round a line singularity.

Because $\epsilon_{\mu ij} n_j$ is real, the superfluid velocity is simply given by the gradient of the phase of the order parameter, i.e.

$$\boldsymbol{v}_s = (\hbar/2m)\boldsymbol{\nabla}\alpha. \tag{11}$$

However, because there are line singularities in the planar phase with α a half-integral multiple of ϕ as well as an integral multiple, circulation of superflow in the planar phase is quantised in half-integral multiples of $\hbar\pi/m$ instead of integral multiples as for the B phase, i.e.

$$\oint \boldsymbol{v}_s \cdot \mathrm{d}\boldsymbol{l} = (\hbar\pi/m)\tfrac{1}{2}N(Z) \tag{12}$$

where $N(Z)$ is the topological quantum number of equation (9).

When the planar phase is confined to a cylindrical container of radius at least one centimetre (so that the surface energy is not swamped by the bending energy), the boundary conditions at the surface of the container may provide a more stringent topological classification of textures. (For a discussion of this point see Bailin and Love (1978a).) At the boundary, $\boldsymbol{n}$ is normal to the surface, i.e.

$$\boldsymbol{n} = \pm\hat{\boldsymbol{\rho}}. \tag{13}$$

The space of order parameters at the boundary is thus

$$A_{\mu i} = \pm e^{i\alpha}\epsilon_{\mu ij}\hat{\rho}_j. \tag{14}$$

Since the minus sign may be absorbed into the phase of the order parameter by replacing α by $\alpha+\pi$, the order parameter space is simply

$$\tilde{R} = S^1 \tag{15}$$

and we may take $\boldsymbol{n} = \hat{\boldsymbol{\rho}}$ for definiteness. The line singularities in a cylindrical container are classified by

$$\pi_1(\tilde{R}) = Z \tag{16}$$

when account is taken of boundary conditions; e.g. the texture

$$(\alpha, \boldsymbol{n}) = (m\phi, \hat{\boldsymbol{\rho}}) \tag{17}$$

may be assigned a topological quantum number m. In the absence of boundary conditions it would have been classified according to equation (9) by $N(Z) = 2m$.

Thus, the effect of the boundary conditions for line singularities is not to introduce any new topological quantum numbers but merely to eliminate the possibility of textures involving half-integral multiples of ϕ.

Non-singular textures in a cylindrical pore may be studied by considering mappings of the cross section of the pore into order parameter space, i.e. of a disc or hemisphere into R with its boundary mapped into $\tilde{R}$ (see Bailin and Love 1978b, Volovik and Mineev 1978). In the present case, the problem is one of mapping a hemisphere to a sphere (associated with $\boldsymbol{n}$) with the boundary of the hemisphere mapped to the equator of the sphere ($\boldsymbol{n} = \hat{\boldsymbol{\rho}}$). The phase factor $e^{i\alpha}$ may be set to a constant because we are dealing with non-singular textures. Such mappings have already been discussed in connection with the B phase by Bailin and Love (1978b). The topologically inequivalent mappings are given in equations (10), (11) and (12) of that paper, and are characterised by an integer N such that the northern hemisphere of the sphere is covered $(N+1)$ times and the southern hemisphere N times. There are also inequivalent textures in which the roles of the northern and southern hemispheres are interchanged.

When a magnetic field $\boldsymbol{H}$ is applied to the planar phase (in bulk) the $\boldsymbol{n}$ vector is aligned parallel or antiparallel to $\boldsymbol{H}$. Let us choose

$$\boldsymbol{H} = H\hat{\boldsymbol{z}}. \tag{18}$$

Then

$$\boldsymbol{n} = \pm\hat{\boldsymbol{z}} \tag{19}$$

and the order parameter space is reduced to

$$A_{\mu i} = \pm e^{i\alpha}\epsilon_{\mu ij}\hat{z}_j. \tag{20}$$

Since the minus sign may be absorbed into the phase factor we see that the order parameter space is

$$R_H = S^1 \tag{21}$$

and

$$\pi_1(R_H) = Z. \tag{22}$$

The topological quantum is simply the winding number for the phase of the order parameter.

A magnetic length ξ_H may be defined such that when the texture varies on this scale of distance the bending energy is comparable with the magnetic energy. Magnetic fields much larger than one gauss must not be applied, otherwise a transition to the A_1 phase occurs. For magnetic fields of a few gauss, $\xi_H \approx 1$ cm, and for lower magnetic fields ξ_H is greater. Non-singular textures such that the magnetic energy is minimised only at the extremities of the texture can occur on the scale of the magnetic length. If the texture is cylindrical we shall refer to it as a cylindrical soliton, and if it is planar as a domain wall.

In the case of cylindrical solitons, a topological classification (Volovik and Mineev 1978) involves mapping a disc, representing the cross section of the cylindrical soliton, onto a sphere (associated with $\boldsymbol{n}$), with the boundary of the disc mapped onto the south pole of the sphere ($\boldsymbol{n} = -\hat{z}$). The phase factor $e^{i\alpha}$ may be set to a constant for a non-singular texture. Since the boundary of the disc is mapped to a single point, the problem is the same as mapping a sphere to a sphere, and inequivalent textures are characterised by an integer q, which is the number of times the second sphere is covered when the first sphere is covered once. Examples of textures with topological quantum number q are

$$\boldsymbol{n} = \hat{z} \cos \beta(\rho/\xi_H) + \sin \beta(\rho/\xi_H)(\hat{x} \cos q\phi + \hat{y} \sin q\phi) \tag{23}$$

with $0 \leqslant \beta \leqslant \pi$ and

$$\begin{aligned} \beta &\to 0 \qquad \text{as} \qquad \rho \to 0 \\ \beta &\to \pi \qquad \text{as} \qquad \rho \to \infty. \end{aligned} \tag{24}$$

Domain walls may be classified by the relative homotopy group $\pi_1(R, R_H)$ which describes mappings of a line into the order parameter space R with its end-points mapped into R_H (Volovik and Mineev 1978). $\pi_1(R, R_H)$ may be computed from an exact sequence of homomorphisms (see, for example, Steenrod 1951):

$$\pi_1(R_H) \xrightarrow{i} \pi_1(R) \to \pi_1(R, R_H) \to \pi_0(R_H),$$

i.e.

$$Z \xrightarrow{i} Z \to \pi_1(R, R_H) \to 0. \tag{25}$$

Using the exactness of the sequence,

$$\pi_1(R, R_H) = \pi_1(R)/\mathrm{i}\pi_1(R_H). \tag{26}$$

Since the mapping i is the inclusion of $\pi_1(R_H)$ in $\pi_1(R)$, we know from our preceding discussions that it maps the integers onto the even integers. Thus

$$\pi_1(R, R_H) = Z_2. \tag{27}$$

An example of a domain wall with $N(Z_2) = 1$ is

$$(\alpha, \boldsymbol{n}) = (\alpha(x/\xi_H), \hat{z} \cos \gamma(x/\xi_H) + \hat{y} \sin \gamma(x/\xi_H)) \tag{28}$$

with

$$\begin{aligned} &\alpha \to 0, \gamma \to 0 \qquad \text{as } x \to -\infty \\ &\alpha \to \pi, \gamma \to \pi \qquad \text{as } x \to +\infty. \end{aligned} \tag{29}$$

Since the space R_H is connected, the ends of any planar phase domain wall must be in the same connected component of R_H. This means that the topological stability of planar phase domain walls could *not* be understood without the use of relative homotopy groups.

The surface energy for the planar phase can also produce domain walls. Between parallel plates separated by about 1 mm these domain walls have length about 1 mm. The topological discussion is exactly analogous to that for magnetic domain walls with the normal to the surface playing the role of the magnetic field. (Similar structures have been described in the B phase by Maki and Kumar (1977).)

We would like to thank D Bailin for a helpful discussion. This research was supported in part by the Science Research Council under grant number GR/A/43087. One of us (SZ) is grateful to the Science Research Council for a research studentship.

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